Quick Clicker Questions:

Question 1: Is the set of even counting numbers smaller than the set of all counting numbers?
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(A) Yes     (B) No.
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(A) Yes    (B) No.
Counting

When we count, we pair off the things we want to count with the counting numbers:

(a,1), (b,2), (c,3), ... (z,26)

It is this association between the counting numbers and the objects we want to count that gives us our first notion of size, or cardinality of sets.

The size of a set of things is simply the number of things:

\{a, b, c, \ldots, z\}

has 26 elements:

|\{a, b, c, \ldots, z\}| = 26

So what is |\{9\}|?
Counting

When we count, we pair off the things we want to count with the counting numbers:

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The size of a set of things is simply the number of things:

\(\{a, b, c, \ldots, z\}\) has 26 elements:

\(|\{a, b, c, \ldots, z\}| = 26\)

So what is \(|\{9^9\}| \)?
The counting numbers are just symbols for the idea they represent. \{\} is the set with no elements, the empty set. It has cardinality 0: \(|\{}\| = 0\).

\{\}\ is the set containing only the empty set. It has cardinality 1: \(|\{\}\| = 1\).

\{\},\ {{}\} contains the empty set and the set containing the empty set. It has cardinality 2: \(|\{\},\ {{}\}\| = 2\).
(a, I), (b, II), (c, III), (d, IV), (e, V), (f, VI) ...

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\{\{\}, \{\{\}\}\}, \{\{\}, \{\{\}\}\}\} \text{contains the empty set, the set containing the empty set, and the set containing the empty set and the set containing the empty set. It has cardinality 3:}

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Want to do 4?
\{ \{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}\} contains the empty set, the set containing the empty set, and the set containing the empty set and the set containing the empty set.

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$$|\{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}| = 3$$

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What about counting the counting numbers?
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Want to do 4?

What about counting the counting numbers?

Let’s pair them off:

(1,1), (2,2), (3,3), ..., (n,n), ...
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Want to do 4?

What about counting the counting numbers?

Let’s pair them off:
(1,1), (2,2), (3,3), ..., (n,n), ...

Does this end?

We say that a set is finite if this pairing off does end.
Counting

What if we lost the number 1?

Is \{2, 3, 4, \ldots\} smaller than \{1, 2, 3, 4, \ldots\}?

Let's count: (1,2), (2,3), (3,4), ...

\[
\begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
\end{array}
\]

We have each counting number associated with exactly one other number and none are missed!

If it helps, we can write down the association:

\[ n \rightarrow n + 1. \]
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\[
\begin{align*}
1 & \rightarrow 2 \\
2 & \rightarrow 3 \\
\vdots & \vdots \\
2 & \rightarrow 3 \\
\end{align*}
\]

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We have each counting number associated with exactly one other number and none are missed!

If it helps, we can write down the association: \( n \rightarrow n + 1 \).
Cardinality

This idea of associating with each counting number exactly one element of a set is fundamental! We say the sets are in one-to-one correspondence. If we have a one-to-one correspondence between some set and the counting numbers then we can count the elements of the set.

Now for the real definition of cardinality: Two sets have the same cardinality if there is a one-to-one correspondence between (all of) the elements of the sets. We don't need infinity to be a number (today), but let's say that infinity is a cardinality. If a set is finite, then its cardinality is the number of elements it contains.
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So removing a single integer doesn’t change the size of the natural numbers...

What about removing half of them?
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What is the cardinality of the even counting numbers?

$$n \rightarrow 2n$$

1 2 3 4 5 6 7 8 9 10 11 ...  
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow  
2 4 6 8 10 12 14 16 18 20 22 ...
Hilbert’s Hotel
Hilbert’s Hotel

Where we are always full...
Hilbert’s Hotel

Where we are always full...

... and we always have rooms available!
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...
And this one-to-one correspondence tells us that the fractions (the rational numbers) are countable:

Cardinality of natural numbers = Cardinality of rational numbers
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